Asymptotically symmetric manifolds and Ricci flow

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Outline

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Hyperbolic space vs conformal round sphere

Let us consider **Hyperbolic space** as a symmetric space

$$\mathrm{SO}^{\uparrow}(1, n+1)/\mathrm{SO}(n+1) = \mathbb{H}^{n+1}$$

and **the conformal round sphere** as the boundary at infinity of $\mathbb{H}^{n+1}$ and a homogeneous space of the parabolic geometry:

$$\mathrm{SO}^{\uparrow}(1, n+1)/\mathcal{H} = \mathbb{S}^{n}$$

where $\mathcal{H}$ is the group generated by rotations, scalings, and inversions on the Euclidean space $\mathbb{R}^n$. The identifications

$$\mathrm{SO}^{\uparrow}(1, n+1) \cong \mathrm{Isom}(\mathbb{H}^{n+1}) \cong \mathrm{Conf}(\mathbb{S}^{n})$$

tell us how the **Lorentz group** acts on the hyperboloid in Minkowski spacetime and how the isometry group acts on the space of classes of equivalent geodesic rays in $\mathbb{H}^{n+1}$.

Hyperbolic space vs conformal round sphere

$(\mathbb{R}^{n+2}, g)$, $g = -dx_0^2 + |d\mathbf{x}|^2 + dx_{n+1}^2$

- **Hyperboloid** $\{(x_0, \mathbf{x}, x_{n+1}) | -x_0^2 + |x|^2 + x_{n+1}^2 = -1\}$ which is a orbit of $SO(1, n+1)$
- **Light cone** $\{(x_0, \mathbf{x}, x_{n+1}) | -x_0^2 + |x|^2 + x_{n+1}^2 = 0\}$ which is a orbit of $SO(1, n+1)$

AHE manifold vs conformal infinity

(X^{n+1}, g^+) is said to be asymptotically hyperbolic (AH in short) when (X^{n+1}, g^+) is conformally compact and, in addition, the sectional curvature goes to -1 at infinity and AH can induce a conformal structure on the boundary $\partial X^{n+1} = M^n$.

We like to impose the Einstein conditions

$$\mathrm{Ric}[g^+] = -ng^+$$

to make the association

$$\partial_\infty(X^{n+1}, g^+) = (M^n, [\hat{g}])$$

possibly canonical, in which (X^{n+1}, g^+) is said to be asymptotically hyperbolic Einstein (AHE in short).

The holographic principles

Given an AHE (X^{n+1}, g^+) and the geodesic defining function x associated with a representative $\hat{g}$ of its conformal infinity $(M^n, [\hat{g}])$, one may consider the volume expansion, for n odd,

$$vol_{g^+}(\{x > \epsilon\}) = v_0\epsilon^{-n} + \cdots + v_{n-1}\epsilon^{-1} + V + o(1)$$

and for n even

$$vol_{g^+}(\{x > \epsilon\}) = v_0\epsilon^{-n} + \cdots + v_{n-2}\epsilon^{-2} + H \log x + V + o(1)$$

Theorem (Henningson-Skenderis 1998 and Graham 1999)

For odd n, the so-called renormalized volume $V = V(X^{n+1}, g^+)$ is independent of the choice of representatives of the conformal infinity.

Theorem (Anderson 2001)

On an AHE (X^4, g^+)

$$8\pi^2\chi(X^4) = \int_X (|W|^2 dvol)[g^+] + 6V(X^4, g^+).$$

AdS/CFT correspondences in Mathematics

Theorem (Li-Qing-Shi 2016)

Suppose that (X^{n+1}, g^+) is AHE with the conformal infinity $(M^n, [\hat{g}])$ of positive Yamabe type. Then, for any $p \in X^{n+1}$,

$$\left(\frac{Y(M^n, [\hat{g}])}{Y(\mathbb{S}^n, [g_{\mathbb{S}}])}\right)^{\frac{n}{2}} \leq \frac{vol(\partial B_{g^+}(p,t))}{vol(\partial B_{g_{\mathbb{H}}}(t))} \leq \frac{vol(B_{g^+}(p,t))}{vol(B_{g_{\mathbb{H}}}(t))} \leq 1$$

where

$$Y(M^n, [g]) = \inf_{g \in [g]} \frac{\int_{M^n} R_g \, d\text{vol}_g}{\text{vol}(M^n, g)^{\frac{n-2}{n}}}$$

is the Yamabe constant.

Corollary

If $(M^n, [\hat{g}])$ is the conformally round sphere, then (X^{n+1}, g^+) is the hyperbolic space.

Outline

Symmetric space

Definition (Symmetric space)

A Riemmaninan manifold (M, g) is called **symmetric space** if for arbitrary point $p \in M$ there exist a a reflection Φ_p at p.

Definition (Symmetric space of noncompact type)

A symmetric space is of noncompact type, if its section curvature is strictly negative.

Theorem (Parallel curvature)

A Riemannian manifold is symmetric space if and only if it is simply connected and $\nabla R \equiv 0$.

De Rham decomposition

Definition (Irreducible symmetric space)

A symmetric space is called irreducible if it can not be decomposed into a product of two symmetric space.

Theorem (De Rham decomposition)

Let M be symmetric space. Then M is a product

$$M = M_1 \times \ldots \times M_r$$

where the factors M_i are irreducible.

Theorem

Irreducible symmetric space is an Einstein manifold.

Semisimple Lie algebra and root decomposition (I)

Let (M, g) be a symmetric space of noncompact with the isometric group G. Then, it turn out that G is semisimple and transitive acting on M. Therefore, if fix a point $p_0 \in M$ and put $K \triangleq \{g \in G | g(p_0) = p_0\}$ (K is called the isotropic group fixing p_0), then symmetric space can be identified as homogeneous space $M \cong G/K$. In fact, G is semisimple.

Let $\mathfrak{g}$ and $\mathfrak{l}$ be the Lie algebra of G and K respectively.

- **The involution on $\mathfrak{g}$** There is the induced involution $\sigma^2 = Id$ from the reflection at p_0.
- **The Cartan decomposition $\mathfrak{g}$** The Lie algebra of $\mathfrak{g}$ can have the following decomposition

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{l}$$

  where $\mathfrak{p}$ and $\mathfrak{l}$ are the eigenspaces of σ for the eigenvalues -1 and 1 respectively. Moreover, $\mathfrak{p} \cong T_{p_0}M$ can be thought of as the infinitesimal of translation.

Semisimple Lie algebra and root decomposition (II)

- **Inner product** $(.,.) \triangleq -<.,\sigma(.)>$ is an inner product on $\mathfrak{g}$ where $<.,.>$ is the Killing form on $\mathfrak{g}$.
- **The rank of symmetric space** The dimension of the maximal Abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ is called the rank of the symmetric space.
- **Root system** We say that $\alpha \in \mathfrak{a}^*$ is a root of $\mathfrak{g}$ relative to $\mathfrak{a}$ if $\alpha \neq 0$ and there exists some common eigenvector $X \neq 0 \in \mathfrak{g}$ such that $[v, X] = \alpha(v)X$ for any $v \in \mathfrak{a}$. Denote Δ the set of all the root. And denote $\mathfrak{g}_\alpha$ the corresponding eigenspace about α.
- **Positive root** There exists $v_0 \in \mathfrak{a}$ such that $\alpha(v_0) \neq 0$ for all nonzero $\alpha \in \Delta$. Therefore, we define the set of positive roots by $\Delta_+ = \{\alpha \in \Delta : \alpha(v_0) > 0\}$.
- **Root system properties** $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ and $\sigma(\Delta_+) = -\Delta_+ \subset \Delta$.
- **Root decomposition** The Lie algebra $\mathfrak{g}$ of G can be decomposed as following

$$\mathfrak{g} = \mathfrak{a} \oplus_{\alpha \in \Delta_+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \oplus \mathfrak{l}_0.$$

Semisimple Lie algebra and root decomposition (III)

- **Frame** Let $\alpha_1, \cdots, \alpha_{n-r}$ be the roots of Δ_+ occurring with the appropriate multiplicities and let $x_1, \cdots, x_{n-r}$ be an orthonormal basis of $\mathfrak{n}_+$ with respect to $(.,.)$ such that $x_i \in \mathfrak{g}_{a_i}$. Then $[x_i, x_j] \in g_{\alpha_i+\alpha_j}$. So $< x_i, y_j >= -\delta_{ij}$ and $< x_i, x_j >= 0$ and $[x_i, y_j] \in \mathfrak{g}_{\alpha_i - \alpha_j}$ and $[y_i, y_j] \in \mathfrak{g}_{-\alpha_i-\alpha_j}$. We set

$$\mathfrak{p}_i = \frac{1}{\sqrt{2}}(x_i - y_i), \quad k_i = \frac{1}{\sqrt{2}}(x_i + y_i)$$

  Hence, $p_1, \cdots, p_{n-r}$ form an orthonormal basis of the orthogonal complement $\mathfrak{a}^{\perp}$ of $\mathfrak{a}$ in $\mathfrak{p}$ and $k_1, \cdots, k_{n-r}$ are a negative orthonormal basis of the orthogonal complement of $\mathfrak{l}_0$ in $\mathfrak{l}$.

- **Curvature** For $X, Y, Z \in \mathfrak{p} \cong T_{p_0}M$, the Riemannian curvature of the symmetric space, M, is

$$R(X,Y)Z|_{p_0} = [Z,[X,Y]]|_{p_0}$$

Iwasawa decomposition

Let

$$A = \exp(\mathfrak{a})$$

be the Abelian subgroup,

$$N = \exp(\oplus_{\alpha\in\Delta_+}\mathfrak{g}_\alpha)$$

be the nilpotent subgroup and

$$K_0 = \exp(\mathfrak{k}_0)$$

be the compact subgroup. Then we have the subgroup

$$P \cong ANK_0$$

which is parabolic.

Outline

Definition of ideal boundary

- **The asymptotic ray** Two (unit speed) geodesics ray $\sigma, \tau : [0, +\infty) \longrightarrow M$ are called asymptotic if the function $t \mapsto d(\sigma(t), \tau(t))$ is bounded.
- **Martin boundary** The boundary at infinity $\partial_\infty M$ of M is the set of equivalence classes of rays.
- **The topology of '$\partial_\infty M$'** Let $U_x M \subset T_x M$ be the unit sphere in $T_x M$. Then the map $\Phi_x : U_x M \longrightarrow \partial_\infty M$ is bijective. Because symmetric spaces of noncompact are Hadamard.

Ideal boundary as homogeneous space

- **The group action on '$\partial_\infty M$'** The isometric group of M, G, take geodesic rays to geodesic rays, which act on the $\partial_\infty M$ transitively.
- **The geometry of the boundary** Let $\xi \in \partial_\infty M$ and G_ξ be the isotropic group at ξ. Then $\partial_\infty M = G/G_\xi$.

Parabolic group

- **The graded Lie algebra** A Lie algebra $\mathfrak{g}$ is graded Lie algebra if there exists a decomposition of $\mathfrak{g} = \oplus_{i=-k}^{k} \mathfrak{g}_i$ such that $[\mathfrak{g}_i, \mathfrak{g}_j] \in \mathfrak{g}_{i+j}$ ($\mathfrak{g}_i = 0$ if $|i| > k$) and the subalgebra $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ can be generated by $\mathfrak{g}_{-1}$
- **The parabolic subgroup** If G is the Lie group with the graded Lie algebra, then $K < G$ is the parabolic subgroup if the Lie algebra of K is the $\oplus_{i=0}^{k} \mathfrak{g}_i$
- **The parabolic geometry** If G is a semisimple Lie group and K is the parabolic subgroup of G. Then the Cartan geomerty $(P \to M, \omega)$ of the type (G, K) is a parabolic geometry.
- **The Levi subgroup** If G is a Lie group with the graded Lie algebra, then $G_0 < G$ is the Levi subgroup if the Lie algebra of G_0 is $\mathfrak{g}_0$. For semisimple Lie group, $G_0 = AK_0$ is the Levi subgroup

Outline

Real hyperbolic space and conformal round sphere (I)

- **Hyperbolic space** is a symmetric space given as $\mathbb{H}^{n+1} = SO(1, n+1)/SO(n+1) = G/K$.
- **The Lie algebra of the Isometric group** $\mathfrak{g} = so(1, n+1) = Span\{$ $E_i = \begin{bmatrix} 0 & e_i \\ e_i^t & 0 \end{bmatrix}$ and $E_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & a_{ij} \end{bmatrix}$ $\}$ where e_i is the ith coordinate vector for $i = 1, 2, \cdots, n+1$, and a_{ij} is an anti-symmetric $(n+1) \times (n+1)$ -matrix whose entries are all zercept 1 at the intersection of i th row and j th column and -1 at the intersection of j th row and ith column, for all $i, j = 1, 2, \cdots, n+1$ and $i < j$.
- **The Killing Form** $< \mathrm{M}, \mathrm{N} >= \mathrm{tr} \mathrm{MN} - \frac{1}{n+2} \mathrm{tr} \mathrm{M} \, \mathrm{tr} \, \mathrm{N}$
- **The Cartan decomposition** $so(1, n+1) = \mathfrak{p} \oplus \mathfrak{l}$ where $\mathfrak{p} = \mathrm{span}\{E_i : i = 1, 2, \cdots, n+1\}$ and $\mathfrak{l} = \mathrm{span}\{E_{ij} : i, j = 1, 2, \cdots, n+1 \text{ and } i < j\}$

Real hyperbolic space and conformal round sphere (II)

- **The maximal abelian algebra** $\mathfrak{a} = \{v_1 = \begin{pmatrix} 0 & e_1 \\ e_1^T & 0 \end{pmatrix}\}$
- **The root decomposition**

$$\mathfrak{g}_{+1} = \text{span}\{E_i + E_{1i} : i = 2, 3, \cdots, n+1\}$$
$$\mathfrak{g}_{-1} = \text{span}\{E_i - E_{1i} : i = 2, 3, \cdots, n+1\}$$
$$\mathfrak{l}_0 = \text{span}\{E_{ij} : i, j = 2, 3, \cdots, n+1 and i < j\}$$

- **The metric of the hyperbolic space under the spherical coordinate**

$$g_0 = dt^2 + \sinh^2(-t) g_{\mathbb{S}}$$

Notice that $g_{\mathbb{S}}$ is the standard metric on the sphere.

Real hyperbolic space and conformal round sphere (III)

- **The parabolic group** $P = AN^+K_0$ where A, N^+, K_0 correspond to the Lie algebra $\mathfrak{a}, \mathfrak{g}_+, \mathfrak{l}_0$ respectively. Moreover, this group fix the infinity in the sense of Martin boundary.
- **The boundary geometry** $M = G/P = \mathbb{S}^n$ and the tangent space of M at P is equivalent to $\mathfrak{g}_{-1}$.
- **Levi group** AK_0 Scalings and rotations of $T_{p_0}M$ and it is the linear automorphism group of the conformal structure on $T_{p_0}M$.

Complex hyperbolic space and standard CR sphere

- **Complex Hyperbolic Space** $\mathbb{CH}^{n+1} = \frac{SU(1,n+1)}{U(n+1)}$
- **The Lie algebra of the Isometric group**
  $su(1,n+1) = \begin{pmatrix} 0 & v_1 \\ v_1^T & A \end{pmatrix} \oplus \sqrt{-1}\begin{pmatrix} a & v_2 \\ -v_2^T & A_S \end{pmatrix}$ where v_1 and v_2 are real row vector, A is an anti-symmetric matrix and A_S is a symmetric matrix. The involution σ between the $SU(1,n+1)$ can induce an
- **Killing form** $K(M,N) = (4n+8)\,\mathrm{Tr}(MN)$
- **Cartan decomposition**
  $\mathfrak{k} = \mathrm{span}\left\{\begin{pmatrix} 0 & 0 \\ 0 & E_{ij} - E_{ji} \end{pmatrix}; \sqrt{-1}\begin{pmatrix} 0 & 0 \\ 0 & E_{ij} + E_{ji} \end{pmatrix}; \sqrt{-1}\begin{pmatrix} -1 & 0 \\ 0 & E_{ii} \end{pmatrix}\right\}$
  $\mathfrak{p} = \mathrm{span}\left\{\begin{pmatrix} 0 & e_i \\ e_i^T & 0 \end{pmatrix}; \sqrt{-1}\begin{pmatrix} 0 & e_i \\ -e_i^T & 0 \end{pmatrix}\right\} \ (i \neq j)$

Complex hyperbolic space and standard CR sphere

- **The root decomposition**

$$\mathfrak{g}_{+1} = \left\{ \begin{pmatrix} 0 & e_i \\ e_i^T & E_{1i} - E_{i1} \end{pmatrix}; \sqrt{-1} \begin{pmatrix} 0 & e_i \\ -e_i^T & E_{1i} + E_{i1} \end{pmatrix} \right\} (i \neq 1)$$

$$\mathfrak{g}_{+2} = \left\{ \sqrt{-1} \begin{pmatrix} -1 & e_1 \\ -e_1^T & E_{11} \end{pmatrix} \right\}$$

$$\mathfrak{l}_0 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & E_{ij} - E_{ji} \end{pmatrix}; \sqrt{-1} \begin{pmatrix} 0 & 0 \\ 0 & E_{ij} + E_{ji} \end{pmatrix}; \right.$$

$$\sqrt{-1} \begin{pmatrix} -1 & 0 \\ 0 & -E_{11} + 2E_{ii} \end{pmatrix} (i \neq 1; j \neq 1; i \neq j)$$

- **Levi group** $U(n)$ and scalings on $T_{p_0}M$ which is the linear automorphism group of the CR contact structure on $T_{p_0}M$.

- **The parabolic group** $P = AN^+K_0$ fixes a point in the Martin boundary. N^+ is the Heisenberg group.

Complex hyperbolic space and standard CR sphere

- **The boundary geometry** $M = G/P = \mathbb{S}^{2n+1}$ and the tangent space of M at P is equivalent to $\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$. (Actually, for each point on G/P, there is such decomposition. Thus, we can think of $\mathfrak{g}_{-2}$ and $\mathfrak{g}_{-1}$ as two distribution on G/P.)
- **The contact form** The contact form is a projection of the tangent space of M

$$\eta : \mathfrak{g}_- \to \mathfrak{g}_{-2} = Im(\mathbb{C})$$

  We can see $Ker(\eta) = \mathfrak{g}_{-1}$ which is a nonintegrable distribution on G/P. Moreover, $d\eta$ is a symplectic form on $ker(\eta)$
- **The complex structure on $\mathfrak{g}_{-1}$** It is easy to see that there is a complex structure on $\mathfrak{g}_{-1}$, J, which is directly induced from the complex structure on $\mathfrak{g}$.
- **The Carnot-Caratheodory metrics** By some computations, $g_V = d\eta(J.,.)$ is a metric on $\mathfrak{g}_{-1}$ called Carnot-Caratheodory metric on $\mathfrak{g}_{-1}$.
- **The metric of the complex hyperbolic space**

$$g_0 = dt^2 + \sinh(-2t)\eta^2 + \sinh(-t)^2 g_V$$

Outline

Natural coordinate of the boundary of symmetric space

- **Real hyperbolic case** We can consider the half plane model

$$\mathbb{R}H^{n+1} = \{(\mathbf{x}, y) | \mathbf{x} \in \mathbb{R}^n, \textit{and } y > 0\}$$

with metric

$$\bar{g} = \frac{1}{y^2}(|dx|^2 + |dy|^2)$$

- **Complex hyperbolic case** We can consider the siegel half plane model

$$\mathbb{C}H^{n+1} = \{(\mathbf{z}, \rho, v) | \mathbf{z} \in \mathbb{C}^n, \rho > 0, v \in \mathbb{R}\}$$

with metric

$$\bar{g} = \frac{4|d\mathbf{z}|^2}{\rho^2} + \frac{1}{\rho^4}(4\rho^2(d\rho)^2 + 4(dv + \mathrm{Im}(\mathbf{z}d\mathbf{z}))^2$$

Asymptotically symmetric manifolds and parabolic boundary at infinity (I)

Definition (Asymptotically hyperbolic case)

Let $\bar{X}^{n+1}$ be a compact manifold with boundary $\partial\bar{X}^{n+1}$. g is a Riemannian metric in the interior of $\bar{X}^{n+1}$. Then, g is called asymptotically hyperbolic metric if for any $p \in \partial\bar{X}^n$, there exist a local coordinate of p, $(U, \varphi, (\mathbf{x}, y))$ where $\mathbf{x} \in \mathbb{R}^n$, $y \in \mathbb{R}$ and $\varphi : U \to (\mathbf{x}, y)$ such that

- $U \cap \partial\bar{X}^{n+1} = \{(\mathbf{x}, y) | y = 0\}$
- $|\varphi^{-1*}(g) - \bar{g}|_{C^{2,\alpha}(\bar{g})} \leq Cy$

where $\bar{g}$ is the standard metric of hyperbolic space, $0 < \alpha < 1$ and C is a constant only depending on the choice of the boundary coordinate $(U, (\mathbf{x}, y))$

There is a natural defining function which can formed by y component of each coordinate. By this defining function we can abstract the boundary structure of $\bar{X}^{n+1}$ by restricting $y^2 g$ on $\partial\bar{X}^{n+1}$. Then we can get a conformal structure on boundary.

Asymptotically symmetric manifolds and parabolic boundary at infinity (II)

Definition (Asymptotically complex hyperbolic case)

Let $\bar{X}^{2n+2}$ be a compact manifold with boundary $\partial\bar{X}^{2n+2}$. g is a Riemannian metric in the interior of $\bar{X}^{2n+2}$. Then, g is called asymptotically complex hyperbolic metric if for any $p \in \partial\bar{X}^n$, there exist a local coordinate of p, $(U, \varphi, (\mathbf{z}, \rho, v))$ where $\mathbf{z} \in \mathbb{C}^n$, $v, \rho \in \mathbb{R}$ and $\varphi : U \to (\mathbf{z}, \rho, v)$ such that

- $U \cap \partial\bar{X}^{2n+2} = \{(\mathbf{z}, \rho, v) | \rho = 0\}$
- $|\varphi^{-1*}(g) - \bar{g}|_{C^{2,\alpha}(\tilde{g})} \leq C\rho$

where $\bar{g}$ is the standard complex hyperbolic metric and C is a constant only depending on the choice of the boundary coordinate $(U, (\mathbf{z}, \rho, v))$

For the complex case, we still have the defining function ρ. The corresponding boundary geometry is the so called CR contact structure.

Asymptotically symmetric Einstein manifolds

If an asymptotically symmetric manifold (M, g) satisfies the Einstein equation

$$Ric(g) - \lambda g = 0$$

then (M, g) is called the Asymptotically symmetric Einstein manifold (ASE in short).

A natural question is given a boundary structure, whether there exists a ASE manifold with this boundary structure?

We have the perturbation result

Theorem (R. Graham, J. Lee 1990 and J. Lee 2006)

Given an AHE manifold $\left(X^{n+1}, g_0^+\right)$, let its conformal infinity be $(M^n, [\hat{g}_0])$. Assume that g_0^+ is non-degenerate. Then, for a C^2 smooth perturbation $[\hat{g}]$ of $[\hat{g}_0]$, there exists an AHE metric g^+ whose conformal infinity is $(M^n, [\hat{g}])$

Asymptotically symmetric Einstein manifolds

Theorem (O.Biquard)

Let $(M, g_0) = \mathbb{K}H^m$ be the complex hyperbolic, quaternionic or octonionic space, respectively, and S its sphere at infinity. Let $V \subset T\mathbb{S}$ be a distribution of codimension $l = 1, 3,$ or 7, respectively. Let η be a 1-form with values in $\mathbb{R}^l$ with kernel V, and γ a Carnot-Caratheodory metric on the distribution V, compatible with $d\eta$. If γ is sufficiently close (in the Holder $C^{2,\alpha}$ norm) to the standard Carnot-Carathodory metric on the boundary of $\mathbb{K}H^m$, then there exists an Einstein metric g on M which is asymptotically symmetric with conformal infinity $[\gamma]$.

Isomorphism theorem (I)

We need to consider the linearization of this equation.

- **Linearization of Einstein equation**

$$[(D_g Ric)h]_{ij} = \frac{d}{dt} Ric(g+th)_{ij}|_{t=0} = -\frac{1}{2}\Delta_L h_{ij} + \frac{1}{2}(\nabla_i V_j + \nabla_j V_i)$$

where h is a symmetric 2-tensor and

$$\Delta_L h_{ij} = \Delta h_{ij} + 2R_{iljk}h^{lk} - R_{il}h^l_j - R_{jl}h^l_i$$

$$V_j = \frac{1}{2}g^{lk}(\nabla_l h_{kj} - \nabla_j h_{lk} + \nabla_k h_{lj})$$

All the ∇ and Δ in the above formula is with respect to g.

Isomorphism theorem (II)

- **Gauge Einstein equation** The following equation is called gauge Einstein equation

$$Ric(g)_{ij} - \frac{1}{2}(\nabla_i W_j + \nabla_j W_i) = \lambda g_{ij}$$

where ∇ is with respect to g and

$$W_j = g^{kl} g_{jb}(\Gamma^b_{kl} - \tilde{\Gamma}^b_{kl})$$

where $\tilde{\Gamma}$ is the christoffel symbol with respect to some metric $\tilde{g}$. The linearization of the gauge Einstein equation is

$$[D_g(Ric_{ij} - \frac{1}{2}(\nabla_i W_j + \nabla_j W_i)](h)_{ij} = -\frac{1}{2}\Delta_L h_{ij}$$

Theorem (J. Lee, R. Graham and O. Biquard)

Let (X, g) be a asymptotically real or complex manifold. If g satisfies the gauge Einstein equation and $|W_j|_g(x)$ goes to zero as $x \in M$ goes to the boundary of M, then g also satisfies the Einstein equation.

Isomorphism theorem of the linearization of gauge Einstein equation is the key to the perturbational existence theorem.

Outline

The Laplacian operator in spherical coordinates (I)

The root system can give us a good frame of symmetric space which is similarto the spherical coordinate. It turn out that the Laplacian operator on thehomogeneous vector bundle of symmetric space has a perfect form in this frame.

- **Principal bundle** The symmetric space have a natural principal bundle structure $\pi : G \to G/K$.
- **Homogeneous vector bundle** The homogeneous vector actually is an associative vector bundle about the associative vector bundle $\pi : G \to G/K$. It turn out that this kind of associative vector bundle can be identified with a group representation of isotropic group K on an vector space E_0, $\rho : K \to GL(E_0)$, by the following way

  $$E = G \times_\rho E_0 = (G \times E_0)/\sim$$

  where $\sim$ is

  $$(gk, v) \sim (g, \rho(k)v)$$

  Moreover, the tangent bundle and symmetric two tensor bundle are all homogeneous vector bundle with the representation

  $$\rho(h) = Ad(h)(\mathfrak{p})$$

  where $h \in K$ and $\mathfrak{p} \cong T_{p_0}M$.

- **Laplacian operator** For $p = \exp(rx_0)(p_0)$, we can write down the Laplacian operator on the frame we took in the previous chapter.

$$\widetilde{\Delta f}|_p = \sum_{j=1}^{r} \nabla_{d\pi_e \widetilde{(p_j)} \nabla_{e\pi_e(x_j)}} f + \frac{1}{sh^2(-\alpha_i(x_0)r)} [\sum_{i=1}^{n-r} \nabla_{d\pi_e \widetilde{(k_i)|_p} \nabla_{d\pi_e(k_i)}} f - \sum_{i=1}^{n-r} \nabla_{\nabla_{d\pi_e \widetilde{(k_i)|_p}} d\pi_e(k_i)} f]$$

$$= \sum_{j=1}^{r} \frac{d^2}{dt^2} \widetilde{f}(\exp(tp_j)\exp(rx_0))|_{t=0} - \sum_{i=1}^{n-r} \frac{ch(-\alpha_i(x_0)r)}{sh(-\alpha_i(x_0)r)} \sum_{j=1}^{r} \alpha_i(p_j) \frac{d}{dt} \widetilde{f}(\exp(tp_j)\exp(rx_0))$$

$$+ \sum_{i=1}^{n} \frac{1}{sh^2(-\alpha_i(x_0)r)} \frac{d^2}{dt^2} \widetilde{f}(\exp(k_i t)\exp(rx_0))|_{t=0}$$

$$+ \sum_{i=1}^{n} \frac{2coth(-\alpha_i(x_0)r)}{sh(-\alpha_i(x_0)r)} \rho_{0*}(k_i) \frac{d}{dt} \widetilde{f}(\exp(k_i t)\exp(rx_0))|_{t=0}$$

$$+ \sum_{i=1}^{n} \frac{ch^2(-\alpha_i(x_0)r)}{sh^2(-\alpha_i(x_0)r)} \rho_{0*}^2(k_i)\widetilde{f}$$

Casimir operators

- **Casimir operator** Here, we use the same notation with the section of root system. Let $\mathfrak{m}_0$ be $\mathfrak{l}_0^{\perp}$ in $\mathfrak{l}$. That is to say $\mathfrak{l} \cong \mathfrak{m}_0 \oplus \mathfrak{l}_0$. (Moreover, we have $[\mathfrak{m}_0, \mathfrak{m}_0] \in \mathfrak{l}_0$ and $[\mathfrak{l}_0, \mathfrak{l}_0] \in \mathfrak{l}_0$) The **Casmir operator** is defined as follow

$$\mathcal{C}(\mathfrak{m}_0, \rho_0) = -\sum_{i=1}^{n-r} \rho_{0*}^2(k_i)$$

  where $\{k_i\}_{i=1}^n$ is orthonormal basis of $\mathfrak{m}_0$ with respect to the Killing form $< .,. >$. This operator does not depend on the choices of the basis $\{k_i\}_{i=1}^n$. $A(r)$ is an $n \times n$ matrix which only depends on r and be thought of as green function of Laplacian operator.
- $\rho_0(h)\mathcal{C}(\mathfrak{m}_0, \rho_0) = \mathcal{C}(\mathfrak{m}_0, \rho_0)\rho_0(h)$ for all $h \in K_0$

Green function of the Laplacian operator

- **Green function** The Green function $G_{\xi_{p_0}}(x) \in \mathrm{Hom}(E_{p_0}, E_x)$ is a section satisfying that

$$(\Delta + \mathcal{R})G_{\xi_{p_0}} = \delta_{p_0}\xi_{p_0}$$

where δ_{p_0} is the Dirac function at $p_0 \in M$ and $\xi_{p_0} \in E_{p_0}$ and $(\Delta + \mathcal{R})$ is the linearization of gauge Einstein equation.

- **The existence of Green function on rank 1 symmetric space of noncompact type**
  - **Weitzenbock formula**

$$\left(\delta^D d^D + d^D \delta^D\right) h = -\Delta h - \mathcal{R}h$$

  - **T. Fujitani inequality for Einstein manifold**

$$< \dot{R}h, h > +S|h|^2/n \leq (n-2)K_{max}|h|^2$$

Theorem

For the rank 1 symmetric space of noncompact type (M, g), the operator $\Delta + \mathcal{R} : H^{2,2} \to L^2$ is an isomorphism.

By this theorem, we see the green function always exists.

Green function of the Laplacian operator

- **The properties of the Green function** The lift of Green function is

$$\widetilde{G}_{\xi_{p_0}}(\exp(rx_0)) = A(r)\xi_{p_0}$$

where $x_0 \in \mathfrak{a}$ and $A(r)$ is a linear transformation of E_0.

  - we have

$$\widetilde{G}_{\xi_{p_0}}(h\exp(rx_0)) = A(r)\rho_0(h^{-1})(\xi_{p_0})$$

  for $h \in K$.

  - The linear transformation $A(r)$ satisfies

$$\rho_0(h)A(r)\rho_0(h^{-1}) = A(r)$$

  where $h_0 = \exp(vt)$ with $v \in \mathfrak{l}_0$ and $t > 0$.

Green function of the Laplacian operator

We can decompose the E_0 into the irreduciable invariant subspace $E_1 \oplus \cdots \oplus E_l$ for the group K_0 which is generated by the lie algebra $\mathfrak{l}_0$.

Since

$$\rho_0(h)A(r) = A(r)\rho_0(h) \quad \mathcal{C}(\mathfrak{m}_0, \rho_0)\rho_0(h) = \rho_0(h)\mathcal{C}(\mathfrak{m}_0, \rho_0)$$

then

$$A(r)|_{E_i} = f_i(r)id_{E_i}, \mathcal{C}(\mathfrak{m}_0, \rho_0)|_{E_i} = \mu_i id_{E_i}$$

Therefore, for $v \in E_i$, we have

$$\widetilde{\Delta G_v}(\exp(rx_0)) = \partial_r^2 f_i(r)v + \mathcal{H}\partial_r f_i(r)v - \mu_i f_i(r)v + \mathcal{R}_i f_i v + B(r)v$$

where $\mathcal{H} = \lim_{r\to\infty} \mathcal{H}(r)$ and $\mathcal{H}(r)$ the mean curvature of the geodesic sphere with radius r, $|B(r)| < O(e^{-\mathcal{H}r})$ and $\mathcal{C}(\mathfrak{m}_0, \rho_0)$ is the so called Casmir operator.

Therefore,

$$|G_v| \sim O(\exp(-(\frac{\mathcal{H}}{2} + \sqrt{\frac{\mathcal{H}^2}{4} + \lambda})r)) \cdot |v|$$

where λ is the smallest eigenvalue of $\mathcal{C}(\mathfrak{m}_0, \rho_0) - \mathcal{R}$

Green function of the Laplacian operator

Consider the operator

$$P = \Delta_0 - \lambda$$

where Δ_0 stands for the scalar Laplacian operator and λ is the smallest eigenvalue of $\mathcal{C}(\mathfrak{m}_0, \rho_0) - \mathcal{R}$. The following lemma just give us the estimate of the fist eigenvalue of P.

Lemma (O. Biquard)

For compact supported smooth function f, we have

$$\left\|e^{-\gamma r} P f\right\|_{L^2(M)} \geq \left(\frac{\mathcal{H}^2}{4} - \gamma^2 + \mu\right) \left\|e^{-\gamma r} f\right\|_{L^2(M)}$$

Green function of the Laplacian operator

Theorem

The operator

$$\Delta + \mathcal{R} : H^2_\delta \to L^2_\delta \quad \textit{and} \quad \Delta + \mathcal{R} : C^{2,\alpha}_\delta \to C^\alpha_\delta$$

is an isomorphism, provided that

$$\frac{\mathcal{H}}{2} - \sqrt{\frac{\mathcal{H}^2}{4} + \lambda} < \delta < \frac{\mathcal{H}}{2} + \sqrt{\frac{\mathcal{H}^2}{4} + \lambda}$$

where λ is the smallest eigenvalue of $\mathcal{C}(\mathfrak{m}_0, \rho_0) - \mathcal{R}$.

Isomrophism on symmetric space of rank 1

Theorem (J. Lee)

Let $\Delta + \mathcal{R} : C^\infty(\mathbb{B}^{n+1}; E) \to C^\infty(\mathbb{B}^{n+1}; E)$ be the Linearization of gauge Einstein equation of the Poincare ball $\mathbb{B}^{n+1}$. If $|\delta - n/2| < \frac{n}{2}$, then the natural extension $P : C^{k+2,\alpha}_\delta(\mathbb{B}; E) \to C^{k,\alpha}_\delta(\mathbb{B}; E)$ is an isomorphism. (r is the number of the copies of the tensor bundle.)

Theorem (O.Biquard)

Let M be a rank 1 noncompact symmetric space E is symmetric two tensor bundle and $\Delta + \mathcal{R} : C^\infty(M, E) \to C^\infty(M, E)$ be the Laplacian operator on a homogeneous bundle E. If $k \geq 0$ and

$$|\delta - \frac{\mathcal{H}}{2}| < \sqrt{\frac{\mathcal{H}^2}{4} + \lambda}$$

where λ is the largest eigenvalue of the corresponding $\mathcal{C}(\mathfrak{m}_0, \rho_0) - \mathcal{R}$. Then, the natural extension $\Delta + \mathcal{R} : C^{k+2,\alpha}_\delta(\mathbb{B}; E) \to C^{k,\alpha}_\delta(\mathbb{B}; E)$ is an isomorphism.

For real hyperbolic case M^{n+1}, $\lambda = 0$ and $\mathcal{H} = n$ which agree with theorem of J.Lee. For complex hyperbolic case M^{2n+2}, $\lambda = 0$ $\mathcal{H} = 2n + 2$

Isomorphism on asymptotically symmetric manifolds

Theorem (O.Biquard)

The $\Delta + \mathcal{R} : C_\delta^{k+2,\alpha} \to C_\delta^{k,\alpha}$ on the symmetric 2-tensor of AH and ACH is Fredholm, provided that

$$|\delta - \frac{\mathcal{H}}{2}| \leq \frac{\mathcal{H}}{2}$$

The kernel and the kernel do not depend on δ in this interval and are equal to the L^2 kernel and cokernel.

Outline

Introduction of Ricci flow

- **Ricci flow** The Rici flow is the geometric evolution equation in which one starts with a smooth Riemanian manifold $(\mathcal{M}^n, g_0)$ and evolves its metric by the equation there exists a smooth metric g in M satisfying

$$\frac{\partial}{\partial t} g = -2\mathrm{Rc}$$

where Rc denotes the Ricci tensor of the metric g.

- **The normalized Ricci flow**

$$\begin{cases} \dfrac{d}{dt} g(t) = -2\left(Ric_{g(t)} + (n-1)g(t)\right) \\ g(0) = g_0 \end{cases}$$

- **The relation to Ricci flow**

$$g^N(t) = e^{-2(n-1)t} g\left(\frac{1}{2(n-1)}\left(e^{2(n-1)t} - 1\right)\right)$$

- **The relation to the Einstein manifold** If the $g(\infty)$ exists, then $g(\infty)$ is a Einstein metric with $Ric(g(\infty)) = -(n-1)g(\infty)$

Stability of symmetric spaces of noncompact type under Ricci flow

Theorem (R. Bamler 2015)

Let $(M, \bar{g})$ be a locally symmetric space of noncompact type which is Einstein of Einstein constant $\lambda < 0$ and assume that the de Rham decomposition of $\tilde{M}$ contains no factors which are homothetic to $\mathbb{H}^n, (n \geq 2)$ or $\mathbb{CH}^{2n}, (n \geq 1)$ Then there is an $\varepsilon > 0$ depending only on $\tilde{M}$ such that if

$$(1-\varepsilon)\bar{g} < g_0 < (1+\varepsilon)\bar{g}$$

and if (g_t) evolves by (1.1), then g_t exists for all time t and as $t \to \infty$ we have convergence $g_t \longrightarrow \bar{g}$ in the pointed Cheeger-Gromov sense, i.e. there is a family of diffeomorphisms Ψ_t of M such that $\Psi_t^* g_t \longrightarrow \bar{g}$ and $\Psi_t \to \Psi_\infty$ in the smooth sense on every compact subset of M

Stability of symmetric spaces of noncompact type under Ricci flow

Theorem (R. Bamler 2015)

Let $(M, \overline{g})$ be either $\mathbb{H}^n$ for $n \geq 3$ or $\mathbb{CH}^{2n}$ for $n \geq 2$, choose a basepoint $x_0 \in M$ and let $r = d(\cdot, x_0)$ denote the radial distance function. There is an $\varepsilon_1 > 0$ and for every $q < \infty$ an $\varepsilon_2 = \varepsilon_2(q) > 0$ such that the following holds: If $g_0 = \overline{g} + h$ and $h = h_1 + h_2$ satisfies

$$|h_1| < \frac{\varepsilon_1}{r+1} \quad \textit{and} \quad \sup_M |h_2| + \left(\int_M |h_2|^q \, dx \right)^{1/q} < \varepsilon_2.$$

Then the normalized Ricci flow exists for all time and we convergence $g_t \longrightarrow \overline{g}$ in the pointed Cheeger-Gromov sense.

Stability of symmetric spaces of noncompact type under Ricci flow

- **The Linearization of normalized Ricci-deTurk flow** $h_{ij}(t,x) = g_{ij}(t,x) - g_{ij}(0,x)$. Then the Ricci-DeTurck flow is equivalent to the following flow

$$\frac{\partial}{\partial t} h_{ij} = \tilde{\Delta} h_{ij} - 2\tilde{R}_{jlil_2} h^{ll_2} - \tilde{R}_{il_2} h_j^{l_2} + \tilde{R}_{jl_2} h_i^{l_2} - 2(n-1)h_{ij} - 2(\tilde{R}_{ij} + (n-1)\tilde{g}_{ij}) + Q_{ij}(t,x)$$

where $\tilde{\Delta}$, $\tilde{R}$ is respect to $\tilde{g}_{ij} = g_{ij}(0,x)$ and

$$Q_{ij}(t,x) = \tilde{g} * \tilde{g}^{-1} * \tilde{\nabla} h * \tilde{\nabla} h + \tilde{g} * \tilde{g}^{-1} * \tilde{\nabla}^2 h * h$$

- The long time existence depends on the estimate of the heat kernel.

Stability of symmetric spaces of noncompact type under Ricci flow

The main ideal is to show that the L^1 norm of the heat kernel of Δ is exponential decay. Once we get this, we can easily get the long time existence and convergence of Ricci-DeTurk flow. We just show the rank 1 case.

- Let λ_L be the smallest eigenvalue of $\mathcal{C}(\mathfrak{m}_0, \rho_0)$.
- The constant λ_{B} : Here we consider all Bochner formulas for sections in E, i.e. expressions

  $$-\triangle = D^* D + \lambda$$

  for some linear first order operator $D : C^\infty(M; E) \to C^\infty(M; E')$ and its formal adjoint $D^* : C^\infty(M; E') \to C^\infty(M; E)$. Let λ_B be the maximum of all such λ Obviously, $\lambda_B \geq 0$, since we always have the trivial Bochner formula $-\Delta = \nabla^* \nabla$. The constant λ_B bounds the L^2 -decay of k_t, i.e. $\|k_t\|_{L^2(M)} \leq Ce^{-\lambda_B t}$ for all $t > 1$ and some $C < \infty$

Stability of symmetric spaces of noncompact type under Ricci flow

Just like Green function, we can show that the heat kernel of the symmetric space of noncompact type of rank 1 is controlled by the heat kernel of some scalar heat equation.

Consider the scalar function operator

$$-L^{\circ} = \Delta - \lambda_L + \sum_{i=1}^{n-r} 2\mu_i \frac{\operatorname{ch} \alpha_i(v) - 1}{\operatorname{sh}^2 \alpha_i(v)} + \sum_{i=1}^{n-r} \operatorname{cth}\left(\alpha_i(v)\right) \partial_{\alpha_i^{\#}}$$

Just like the Green fundtion we can Let $K_t(v)$ be the matrix corresponding to the heat kernel on the symmetric 2-tensor bundle $v \in \mathfrak{p}$. $K_t(v)$ is semipositive definite matrix. Let $K_t(v)$
And it turn out the heat kernel $K_t(v)(max)$ satisfies that

$$[\partial_t + L^0](K_t(v)(max)) \leq 0$$

Stability of symmetric spaces of noncompact type under Ricci flow

Theorem (R.Bamler 2015)

Let M be of rank 1 and let λ_L and λ_B be defined as above. Then there are constants $c > 0, C < \infty$ such that: If $\lambda_B > \lambda_L$, then

$$ce^{-\lambda_L t} < \|k_t\|_{L^1(M)} < Ce^{-\lambda_L t} \quad \text{for all} \quad t > 0$$

If $\lambda_B < \lambda_L$, then we have at least

$$ce^{-\lambda_L t} < \|k_t\|_{L^1(M)} < Ce^{-\lambda_B t} \quad \text{for all} \quad t > 0$$

Finally, if $\lambda_B = \lambda_L$, the upper bound still holds with λ_B replaced by any $\lambda < \lambda_B$ (where C depends on λ). More precisely, we have

$$ce^{-\lambda_L t} < \|k_t\|_{L^1(M)} < C(\log(t+2))^{1/2}(t+2)^{a/2}e^{-\lambda_L t}$$

where $a = \max\left\{\left(\sum_{i=1}^{n-r} |\alpha_i|\right) / \min_i |\alpha_i| \,, 2\right\}$

Ricci flow on asymptotically symmetric manifolds and convergence to Einstein at time infinity

Theorem (J.Qing, Y.Shi, J.Wu 2011)

Let $(\mathcal{M}^n, g)$, $n \geq 5$, be a conformally compact Einstein manifold of regularity C^2 with a smooth conformal infinity $(\partial\mathcal{M}, [\hat{g}])$. And suppose that the non-degeneracy of g satisfies

$$\sqrt{\lambda} > \frac{n-1}{2} - 2$$

Then, for any smooth metric $\hat{h}$ on $\partial\mathcal{M}$, which is sufficiently $C^{2,\alpha}$ close to some $\hat{g} \in [\hat{g}]$ for any $\alpha \in (0,1)$, there is a conformally compact Einstein metric on $\mathcal{M}$ which is of C^2 regularity and with the conformal infinity $[\hat{h}]$.

Our improvement

In the perturbation result of J.Lee, we only require that $\lambda > 0$ and $n \geq 4$. However in the above result, $n \neq 4$ and for n is larger, λ can not be small. The reason that λ can not be small if n is larger is that we can not easily get the long time existence of Ricci flow on asymptotically hyperbolic manifolds from the method of [JYJ]. However, we can make use of the exponential decay of semi group mimic the Bamler's argument to get a stronger existence of Ricci flow on asymptotically hyperbolic manifolds to overcome this difficulty.

The exponential decay of semigroup is not difficult to get from the previous discussion. The difficult part is that the previous method only works for the linearization of the Ricci-deTurk flow at an Einstein manifold. But if the manifold is asymptotically hyperbolic manifold, we can not easily linearize it. In order to overcome this, we linearized the equation point by point.

Our improvement

Definition

(Resolvent set) We say a real number λ belongs to $\rho(A)$, the resolvent set of A, provided the operator

$$\lambda I - A :\to X$$

is on to one and onto. And if $\lambda \in \rho(A)$, the resolvent operator $R_\lambda : X \to X$ is defined by $R_\lambda u := (\lambda I - A)^{-1} u$

Theorem (Hille-Yosida)

Let A be a closed, densely defined linear operator on X. Then A is the generator of a semigroup $\{S(t)\}_{t\geq 0}$ if and only if

$$(c, \infty) \subset \rho(A) \quad \textit{and} \quad \|R_\lambda\| \leq \frac{1}{\lambda - c} \textit{ for } \lambda > 0$$

Moreover, we have $||S(t)|| \leq e^{-ct}$

Our improvement

$X = C^{0,\alpha}_\delta(Sym^2 T^* \mathcal{M}^n)$ with $\delta \in (0, n)$ and trivial L^2 kernel of P on $Sym^2 T^* \mathcal{M}^n$ is nondegenerate. By the lemma of John Lee [JYJ], the $P = \Delta_L + 2(n-1)Id$ is an isomorphism from $C^{2,\alpha}_\delta$ to $C^{0,\alpha}_\delta$. Then we have

$$||Pu||_{C^{0,\alpha}_\delta} \geq c||u||_{C^{0,\alpha}_\delta}$$

where $c > 0$. And for $c \geq -\lambda$, we have

$$||Pu + \lambda u||_{C^{0,\alpha}_\delta} \geq (\lambda + c)||u||_{C^{0,\alpha}_\delta}$$

Therefore,

$$(-c, \infty) \subset \rho(A) \quad \text{and} \quad \|R_\lambda\| \leq \frac{1}{\lambda + c} \text{ for } \lambda > 0$$

Therefore, P is a generator of a semigroup $S(t)$ with $|S(t)| \leq e^{-ct}$

Improvement

Let $g_{ij}^K(t) = g_{ij}(t + Kl)$ where $g_{ij}(t)$ is the solution of the Ricci-DeTurck flow with $g(0) = g_+$ and K is an positive integer.The following flow is called normalized difference Ricci-DeTurck flow

$$\begin{aligned}\frac{\partial}{\partial t}(g^K - g^{K-1}) = & -2(Ric(g^K) + (n-1)g^K) + 2(Ric(g^{K-1}) + (n-1)g^{k-1}) \\ & + (\nabla_i^K V_j^K + \nabla_j^K V_i^K) - (\nabla_i^{K-1} V_j^{K-1} + \nabla_j^{K-1} V_i^{K-1})\end{aligned}$$

where ∇^K is respect to g^K and $V_j^K = g^{K,ll_1} g_{jk}^K (\Gamma_{ll_1}^k (g^K(t)) - \Gamma_{ll_1}^k (g(0)))$

Improvement

We have the following stronger long time existence and convergence of Ricci-deTurk flow.

Lemma

Let (M^n, g_+) be an asymptotically hyperbolic space, $n \geq 4$, with nondegeneracy $\lambda > 0$ and regularity $C^{2,\alpha}$. Then, for any $\delta \in (0, n-1)$, there exists $\epsilon_0(\lambda) > 0$ $L > 0$, such that if $|Ric(g(0)) + (n-1)g(0)| \leq \epsilon_0 e^{-\delta d(x_0,x)}$, the solution of the normalized Ricci-DeTurck flow $g(t,x)$ has long time existence and converges to an Einstein manifold in the sense of C^2_δ norm. Moreover, the limit metric is an Asymptotically Einstein metric with the same conformal infinity.

With this lemma, we can fully recover the theorem of J.Lee.

Thank you for your time!